Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 339 (2008) 10–17

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Products of square-zero operators

Nika Novak

Department of Mathematics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia

Received 10 November 2006

Available online 28 June 2007

Submitted by G. Corach

Abstract

We characterize matrices that can be written as a product of two or three square-zero matrices. We also consider the same questions for (bounded) operators on an infinite-dimensional, separable, complex Hilbert space and in the Calkin algebra.

Keywords: Square-zero operators; Nilpotent operators; Factorization

1. Introduction

Given a vector space $\mathcal{V}$, a linear endomorphism T on $\mathcal{V}$ is called nilpotent if $T^n = 0$ for some positive integer n. The smallest such integer is called index of nilpotency. If $T^2 = 0$ we say that the endomorphism is square-zero.

Which endomorphisms can be written as a product of two nilpotents? The finite-dimensional case has been studied separately by Laffey [5], Sourour [7] and Wu [9]. They showed that every singular $n \times n$ matrix over an arbitrary field, with exception of 2×2 nonzero nilpotent, can be written as a product of two nilpotent matrices.

In [2], Drnovšek, Müller and Novak considered the operators (i.e., bounded linear transformations) on a complex, separable, infinite-dimensional Hilbert space that can be written as a product of two nilpotent operators. They showed that an operator T is a product of two nilpotent operators if and only if $\dim\ker T = \dim\ker T^* = \infty$. They also showed that if an operator is a product of two nilpotent operators, it is already a product of two nilpotent operators with index of nilpotency at most 3.

In this paper, our focus will be upon the products of square-zero endomorphisms. In 1994, Sullivan [8] found the following characterization for linear transformations on an infinite-dimensional vector space.

Theorem 1.1. *Let T be a linear transformation on an infinite-dimensional vector space $\mathcal{V}$. The following assertions hold*:

(a) *T is a product of two square-zero linear transformations if and only if the codimension of* $\operatorname{im} T \cap \ker T$ *in* $\ker T$ *equals the dimension of $\mathcal{V}$.*
(b) *If* $\dim(\ker T) = \infty$ *and* $\operatorname{codim}(\operatorname{im} T) = \infty$, *then T can be written as a product of three square-zero linear transformations.*

E-mail address: nika.novak@fmf.uni-lj.si.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.06.030

In Section 2, we start by studying products of square-zero matrices on finite-dimensional space. We show that a $n \times n$ matrix T can be written as a product of two square-zero matrices if and only if $\dim(\ker T \ominus (\ker T \cap \operatorname{im} T)) \geqslant \operatorname{rank}(T)$. We also prove that T can be expressed as a product of three square-zero matrices if and only if $\operatorname{rank}(T) \leqslant \frac{n}{2}$. In Section 3 we find analogous results for (bounded) operators on infinite-dimensional Hilbert space. Finally, in Section 4 we consider similar questions in the Calkin algebra.

2. Finite-dimensional case

Let $\mathbb{F}$ be an algebraically closed field. By $M_n(\mathbb{F})$ we denote the algebra of all $n \times n$ matrices over $\mathbb{F}$. If $N \in M_n(\mathbb{F})$ is a square-zero matrix, then the rank of N is less than or equal to $\frac{n}{2}$. So a necessary condition for a matrix $T \in M_n(\mathbb{F})$ to be a product of two or more square-zero matrices is that $\operatorname{rank}(T) \leqslant \frac{n}{2}$. Since $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is not a product of two nilpotent matrices (see, e.g. [7]), the condition is not sufficient. The following result shows that it is a sufficient condition for a matrix to be a product of three square-zero matrices.

Proposition 2.1. *A matrix $T \in M_n(\mathbb{F})$ is a product of three square-zero matrices if and only if* $\operatorname{rank}(T) \leqslant \frac{n}{2}$.

Proof. Suppose that $\operatorname{rank}(T) \leqslant \frac{n}{2}$. By the Jordan canonical form, T is similar to $T_1 \oplus \cdots \oplus T_m \oplus 0$, where each T_i is a direct sum of a Jordan block J_i corresponding to the eigenvalue λ_i and zero matrix, such that the size of T_i is $2\operatorname{rank}(T_i)$. We only need to show that each matrix T_i can be written as a product of three square-zero matrices.

Let us first consider the case where $\lambda_i \neq 0$. Then $T_i = J_i \oplus 0$, where both blocks are of the same size. Then we can write

$$\begin{pmatrix} J_i & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & J_i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -I & -I \\ I & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}.$$

Thus T_i is a product of three square-zero matrices.

Suppose now that $\lambda_i = 0$. If J_i is of the size k, then the zero matrix is of the size $k-2$, since $\operatorname{rank}(T_i) = k-1$. For $k = 2$ the following factorization can be found in [7]:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

For $k > 2$ we can write

$$T_i = \begin{pmatrix} E & E_{(k-1),1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -I & -I \\ I & I \end{pmatrix} \begin{pmatrix} E_{12} & 0 \\ E - E_{12} & E_{(k-1),1} \end{pmatrix},$$

where E_{ij} is a matrix of the size $k-1$ with 1 on the position (i, j) and E a matrix of the size $k-1$ with 1 on first upper diagonal. Since all matrices in the factorization are square-zero, the proof is complete. □

Let us now characterize a product of two square-zero matrices.

Theorem 2.2. *Let $T \in M_n(\mathbb{F})$. The following assertions are equivalent*:

(a) *T is a product of two square-zero matrices.*
(b) $\dim(\ker T \ominus (\ker T \cap \operatorname{im} T)) \geqslant \operatorname{rank}(T)$.
(c) *T is similar to a matrix S such that* $\dim(\ker S \cap \ker S^t) \geqslant \operatorname{rank}(S)$*, where S^t denotes the transpose of S.*

Proof. Let $T = MN$, where M and N are square-zero matrices. Then $\operatorname{im} T \subseteq \ker M$, $\operatorname{im} N \subseteq \ker T$, and $\operatorname{im} T = M(\operatorname{im} N)$. Consider the following decomposition of the space $\mathcal{H}_1 = \ker T \cap \operatorname{im} T$, $\mathcal{H}_2 = \ker T \ominus \mathcal{H}_1$, and $\mathcal{H}_3 = (\ker T)^\perp$. It follows that $\operatorname{im} N \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{H}_1 \subseteq \ker M$, so that $\operatorname{im} T \subseteq M(\mathcal{H}_2)$. Therefore $\operatorname{rank}(T) = \dim(\operatorname{im} T) \leqslant \dim \mathcal{H}_2$. Hence (b) follows from (a).

Assume now that (b) holds. One can verify that if matrices S and T are similar, then $\dim(\ker S \ominus (\ker S \cap \operatorname{im} S)) = \dim(\ker T \ominus (\ker T \cap \operatorname{im} T))$. Therefore $\dim(\ker S \ominus (\ker S \cap \operatorname{im} S)) \geqslant \operatorname{rank}(S)$ for every S similar to T. If S is the Jordan canonical form for matrix T, $\ker S \ominus (\ker S \cap \operatorname{im} S) = \ker S \cap \ker S^t$, and so the assertion (c) holds.

It remains to prove that (c) implies (a). To prove this, it is sufficient to show that S can be written as a product of two square-zero matrices. Let $\mathcal{H}_1$ be a subspace of $\ker S \cap \ker S^t$ with $\dim \mathcal{H}_1 = \operatorname{rank}(S)$, $\mathcal{H}_2 = \ker S \ominus \mathcal{H}_1$ and $\mathcal{H}_3 = (\ker S)^{\perp}$. We note that $\dim \mathcal{H}_3 = \operatorname{rank}(S)$. Hence the matrix of S relative to this decomposition is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A \\ 0 & 0 & B \end{pmatrix}.$$

Define matrices M and N by

$$M = \begin{pmatrix} 0 & 0 & 0 \\ A & 0 & 0 \\ B & 0 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly M and N are square-zero matrices and $S = MN$, which completes the proof. □

Note that similarity does not preserve the inequality in part (c). To show this, let $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Since $\dim(\ker T \cap \ker T^t) = 1$, the inequality holds. For every $x \in \mathbb{F} \setminus \{0\}$ the matrix $S = \left(\begin{smallmatrix} 1 & x \\ 0 & 0 \end{smallmatrix}\right)$ is similar to T, but $\dim(\ker S \cap \ker S^t) = 0$.

3. Infinite-dimensional case

In this section we prove analogous results on the infinite-dimensional space. In Theorem 1.1 Sullivan gave characterizations of linear transformations on a vector space that can be written as a product of two or three square-zero linear transformations. His results are analogues to Proposition 2.1 and the equivalence of (a) and (b) in Theorem 2.2. In this section we will find analogous results for operators (i.e., bounded linear transformations) on a Hilbert space.

Let $\mathcal{H}$ denote a complex, separable, infinite-dimensional Hilbert space. Denote by $\mathcal{B}(\mathcal{H})$ the algebra of all operators on $\mathcal{H}$. Given a square-zero operator N, both $\ker N$ and $\ker N^*$ are infinite-dimensional spaces. It follows easily that a necessary condition for an operator T to be a product of two or more square-zero operators is $\dim \ker T = \dim \ker T^* = \infty$. We begin with an example, which shows that this condition is not sufficient for an operator to be a product of two square-zero operators.

Example 3.1. Let T be an operator on $\mathcal{H} \oplus \mathcal{H}$ of the form $\left(\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right)$. Suppose that T is a product of two square-zero operators M and N. It follows from $MT = 0$ and $TN = 0$ that $\operatorname{im} T \subseteq \ker M$ and $\operatorname{im} N \subseteq \ker T$. Since $\ker T = \operatorname{im} T$, we get $\operatorname{im} N \subseteq \ker M$. Therefore $MN = 0$. This contradicts the fact that $T = MN$.

In this case T is already square-zero operator. In similar way we could see that also the operator $T \oplus A$, where A is an invertible operator, can not be written as a product of two square-zero operators.

Similarly as in finite-dimensional case the necessary condition for an operator to be a product of square-zero operators is also a sufficient condition for an operator to be a product of three square-zero operators.

Theorem 3.2. *An operator $T \in \mathcal{B}(\mathcal{H})$ is a product of three square-zero operators if and only if* $\dim \ker T = \dim \ker T^* = \infty$.

Proof. We only need to prove that the condition is sufficient. Suppose that $\dim \ker T = \dim \ker T^* = \infty$. We can choose a decomposition of $\mathcal{H}$ as a direct sum of infinite-dimensional closed subspaces $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ such that $\mathcal{H}_1 \subseteq \ker T$ and $\mathcal{H}_3 \subseteq \ker T^*$. Then the matrix of T relative to this decomposition is of the form

$$\begin{pmatrix} 0 & A & B \\ 0 & C & D \\ 0 & 0 & 0 \end{pmatrix}.$$

Since all infinite-dimensional closed subspaces of $\mathcal{H}$ are isomorphic to $\mathcal{H}$, there exists a unitary operator $(U \;\; V)$ from $\mathcal{H} \oplus \mathcal{H}$ to $\mathcal{H}$. Then there exist operators X and Y such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} (U \quad V).$$

Define operators M, N and P on $\mathcal{H}$ by

$$M = \begin{pmatrix} 0 & 0 & X \\ 0 & 0 & Y \\ 0 & 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0 & U & V \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to verify that $T = MNP$ and that M, N and P are square-zero operators. $\square$

Let us now consider which operators can be written as a product of two square-zero operators. We can find a necessary condition that is analogue to the assertion (b) in Theorem 2.2. But it is not clear if this is also a sufficient condition. What we can show is that the analogue to the assertion (c) is a sufficient condition.

Proposition 3.3. *If an operator $T \in \mathcal{B}(\mathcal{H})$ is a product of two square-zero operators, then* $\dim(\ker T \ominus (\ker T \cap \overline{\operatorname{im} T})) = \infty$.

Proof. If $\dim(\overline{\operatorname{im} T}) < \infty$, the assertion obviously holds, since $\dim(\ker T) = \infty$. Let us now assume that $\dim(\overline{\operatorname{im} T}) = \infty$. If T is a product of two square-zero matrices, we have, similarly as in the proof of implication from (a) to (b) in Theorem 2.2, $\dim(\ker T \ominus (\ker T \cap \overline{\operatorname{im} T})) \geqslant \dim(\overline{\operatorname{im} T})$. This implies the desired conclusion. $\square$

Proposition 3.4. *If an operator $T \in \mathcal{B}(\mathcal{H})$ is similar to an operator $S \in \mathcal{B}(\mathcal{H})$ with* $\dim(\ker S \cap \ker S^*) = \infty$, *then T is a product of two square-zero operators.*

Proof. It is sufficient to show that S is a product of two square-zero operators. We can choose a decomposition of $\mathcal{H}$ as a direct sum of infinite-dimensional closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H}_1 \subseteq \ker S \cap \ker S^*$. The matrix of S relative to this decomposition is of the form $\left(\begin{smallmatrix} 0 & 0 \\ 0 & A \end{smallmatrix}\right)$. Define operators M and N on $\mathcal{H}$ by

$$M = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}.$$

It is evident that $S = MN$ and that M and N are square-zero operators. $\square$

The following example shows that in general a necessary condition and a sufficient condition are not equivalent. We can show the equivalence if T fulfills an additional condition.

Example 3.5. Let $\mathcal{M}$ and $\mathcal{N}$ be infinite-dimensional closed subspaces of $\mathcal{H}$ with trivial intersection and suppose that $\mathcal{M} + \mathcal{N}_0$ is not closed for every infinite-dimensional subspace $\mathcal{N}_0$ of $\mathcal{N}$. Choose the operators M and N so that $\mathcal{M} = \ker M = \operatorname{im} M$ and $\mathcal{N} = \ker N = \operatorname{im} N$. Hence M and N are square-zero operators. Let $T = MN$. Then $\ker T = \mathcal{N}$, $\operatorname{im} T = \mathcal{M}$ and therefore $\ker T + \overline{\operatorname{im} T} = \mathcal{M} + \mathcal{N}$ is not closed.

Suppose that $T = PSP^{-1}$ for an invertible operator P and an operator S with $\dim(\ker S \cap \ker S^*) = \infty$. Define $\mathcal{N}_0 = P(\ker S \cap \ker S^*)$. Then $\dim \mathcal{N}_0 = \infty$. It is easy to see that $\mathcal{N}_0 \subseteq \mathcal{N}$ and $\mathcal{N}_0 \cap \mathcal{M} = \{0\}$. Since the spaces $P^{-1}(\mathcal{N}_0)$ and $\overline{\operatorname{im} S}$ are orthogonal, the space $P^{-1}(\mathcal{N}_0) + \overline{\operatorname{im} S}$ is closed and therefore $\mathcal{N}_0 + \mathcal{M}$ is closed, which is a contradiction.

Proposition 3.6. *An operator $T \in \mathcal{B}(\mathcal{H})$ is similar to an operator $S \in \mathcal{B}(\mathcal{H})$ with* $\dim(\ker S \cap \ker S^*) = \infty$ *if and only if there exists an infinite-dimensional closed subspace $\mathcal{N}$ of* $\ker T$ *such that $\mathcal{N} \cap \overline{\operatorname{im} T} = \{0\}$ and the space $\mathcal{N} + \overline{\operatorname{im} T}$ is closed.*

Proof. Suppose that T is similar to an operator S with $\dim(\ker S \cap \ker S^*) = \infty$. As in example above we can show that $\mathcal{N} = P(\ker S \cap \ker S^*)$ satisfies all the conditions.

To prove the converse, let $\mathcal{H}_1 = \ker T^*$ and $\mathcal{H}_2 = \overline{\operatorname{im} T}$. Then $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. The matrix of T relative to this decomposition is $\left(\begin{smallmatrix} 0 & 0 \\ A & B \end{smallmatrix}\right)$. The kernel of T is the set of pairs $(x_1, x_2) \in \mathcal{H}_1 \oplus \mathcal{H}_2$ satisfying the condition $Ax_1 + Bx_2 = 0$. Let P denote the orthogonal projection onto $\mathcal{H}_1$. Since $\mathcal{N} + \overline{\operatorname{im} T}$ is a closed space, the subspace $\mathcal{H}_{12} = P(\mathcal{N})$ is also closed by [6, Theorem 2.1]. Since $\mathcal{N}$ and $\mathcal{H}_2$ have trivial intersection, P is injective on $\mathcal{N}$. Therefore $\mathcal{H}_{12}$ is infinite-dimensional.

With respect to the decomposition $\mathcal{H} = \mathcal{H}_{11} \oplus \mathcal{H}_{12} \oplus \mathcal{H}_2$, where $\mathcal{H}_{11} = (\mathcal{H}_{12} \oplus \mathcal{H}_2)^{\perp}$, the matrix of T is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_1 & A_2 & B \end{pmatrix}.$$

We now claim that $\operatorname{im} A_2 \subseteq \operatorname{im} B$. Indeed, if $x \in \mathcal{H}_{12}$ there exists $y \in \mathcal{H}_2$ such that $(0, x, -y) \in \mathcal{N} \subseteq \ker T$. Hence $A_2 x = By$, which proves our claim. It follows from the well-known theorem of Douglas [1] that there exists an operator X such that $A_2 = BX$. Define an operator V on $\mathcal{H} = \mathcal{H}_{11} \oplus \mathcal{H}_{12} \oplus \mathcal{H}_2$ as

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & X & I \end{pmatrix},$$

and let $S = VTV^{-1}$. Since the matrix of S is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_1 & 0 & B \end{pmatrix}$$

and $\mathcal{H}_{12}$ is an infinite-dimensional space contained in $\ker S \cap \ker S^*$, the proof is complete. □

Combining Propositions 3.3, 3.4 and 3.6 we obtain the main result of this section.

Theorem 3.7. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is an operator such that $\ker T + \overline{\operatorname{im} T}$ is closed. Then the following assertions are equivalent*:

(a) *The operator T is a product of two square-zero operators.*
(b) $\dim(\ker T \ominus (\ker T \cap \overline{\operatorname{im} T})) = \infty$.
(c) *The operator T is similar to an operator S with* $\dim(\ker S \cap \ker S^*) = \infty$.

4. The Calkin algebra

In the last section we consider the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, where we denote by $\mathcal{K}(\mathcal{H})$ the ideal of all compact operators. The canonical map from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is denoted by π. The essential spectrum $\sigma_e(T)$, the left essential spectrum $\sigma_{\mathrm{le}}(T)$ and the right essential spectrum $\sigma_{\mathrm{re}}(T)$ of an operator $T \in \mathcal{B}(\mathcal{H})$ is defined as the spectrum, the left spectrum and the right spectrum of $\pi(T)$ in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, respectively.

The image of an operator in the Calkin algebra is nilpotent if and only if some power of the operator is compact. We first consider a necessary condition for an operator to be a product of operators whose images in the Calkin algebra are nilpotents. If $T = MN$, where $\pi(M)$ and $\pi(N)$ are nilpotents, then neither left nor right inverse of $\pi(T)$ exists, and so $0 \in \sigma_{\mathrm{le}}(T) \cap \sigma_{\mathrm{re}}(T)$. We say that T is not semi-Fredholm operator.

The following theorems characterize the operators that can be written as a product of operators whose images in the Calkin algebra are nilpotent elements.

Theorem 4.1. *Let T be an operator on $\mathcal{H}$. The following assertions are equivalent*:

(a) *The operator T is not a semi-Fredholm operator.*
(b) *$\mathcal{H}$ can be decomposed as a direct sum of three infinite-dimensional closed subspaces, so that T is similar to an operator of the form*

$$\begin{pmatrix} 0 & A & 0 \\ K & C & D \\ 0 & L & 0 \end{pmatrix},$$

where K and L are compact operators.
(c) *The operator T is a product of two quasi-nilpotent operators.*
(d) *The operator T is a product of two quasi-nilpotent operators M and N such that M^3 and N^3 are compact.*

(e) *The operator T is a product of two operators M and N such that M^3 and N^3 are compact.*
(f) *The operator T is a product of three operators whose squares are compact.*

Proof. For the reader's benefit we sketch the proof of implications from (a) to (b) and from (b) to (c). The detailed proofs can be found in [2]. Suppose that T is not semi-Fredholm. Since neither T nor T^* is upper semi-Fredholm, we can find inductively an orthonormal sequence $f_1, g_1, f_2, g_2, \dots$ in $\mathcal{H}$ such that $\|Tf_n\| \leqslant \frac{1}{n}$ and $\|T^* g_n\| \leqslant \frac{1}{n}$.

Let $\mathcal{M}$ be the closed linear span of the vectors $\{f_n\}$, and let $\mathcal{N}$ be the closed linear span of $\{g_n\}$. Then $\mathcal{H} = \mathcal{M} \oplus \mathcal{L} \oplus \mathcal{N}$, where $\mathcal{L} = (\mathcal{M} \oplus \mathcal{N})^\perp$ (we can assume that $\mathcal{L}$ is also infinite-dimensional). The matrix of T relative to this decomposition is

$$\begin{pmatrix} K_1 & A & B \\ K_2 & C & D \\ K_3 & K_4 & K_5 \end{pmatrix},$$

where $K_1, \dots, K_5$ are Hilbert–Schmidt operators, and hence compact. It can be showed that T is similar to an operator with infinite-dimensional zeros in all four corners. So (a) implies (b).

Since all three spaces in the decomposition are infinite-dimensional, we can write the first and the last of them as an infinite sum of Hilbert spaces that are isomorphic to $\mathcal{H}$ so that in the obtained decomposition of the space $\mathcal{H}$ the operator T can be represented as

$$\begin{pmatrix}
 & & \vdots & \vdots & \vdots & & \\
 & & 0 & A_2 & 0 & & \\
\cdots & 0 & 0 & A_1 & 0 & 0 & \cdots \\
\cdots & K_2 & K_1 & \boxed{C} & D_1 & D_2 & \cdots \\
\cdots & 0 & 0 & L_1 & 0 & 0 & \cdots \\
 & & 0 & L_2 & 0 & & \\
 & & \vdots & \vdots & \vdots & &
\end{pmatrix},$$

where $\{K_n\}$ and $\{L_n\}$ are compact operators satisfying $\max\{\|K_n\|, \|L_n\|\} \leqslant 4^{-n}$ for all $n \geqslant 2$. Now define the operators Q_1 and Q_2 on $\mathcal{H}$ by

$$Q_1 = \begin{pmatrix}
 & & & & \vdots & \vdots & \vdots & & \\
 & & & & 0 & A_3 & 0 & & \\
 & & & & 0 & A_2 & 0 & & \\
 & & & & 0 & A_1 & 0 & 0 & \cdots \\
\cdots & 2^{-2}I & 2^{-1}I & I & \boxed{0} & C & 0 & 0 & \cdots \\
 & & & & 0 & 0 & 2L_1 & & \\
 & & & & 0 & & 0 & 2^2 L_2 & \\
 & & & & 0 & & & 0 & \ddots \\
 & & & & \vdots & & & & \ddots
\end{pmatrix}$$

and

$$Q_2 = \begin{pmatrix}
\ddots & \ddots & & & \vdots & \vdots & & & \\
 & 0 & 2^2 K_2 & & 0 & 0 & & & \\
 & & 0 & 2K_1 & 0 & 0 & 0 & 0 & \cdots \\
 & & & 0 & 0 & D_1 & D_2 & D_3 & \cdots \\
\cdots & 0 & 0 & 0 & \boxed{0} & 0 & 0 & 0 & \cdots \\
 & & & & I & & & & \\
 & & & & 2^{-1}I & & & & \\
 & & & & 2^{-2}I & & & & \\
 & & & & \vdots & & & &
\end{pmatrix}.$$

We can show that Q_1 and Q_2 are quasi-nilpotent operators, so the assertion (c) follows. Since third powers of both operators in the factorization are compact, this also implies (d). Besides, the implication from (d) to (e) is trivial. On the other hand, the necessary condition for an operator to be a product of operators whose images in Calkin algebra are nilpotents shows that each of the assertions (d)–(f) implies (a).

Finally, let us show that (b) implies (f). Suppose that T is similar to an operator S of the form

$$\begin{pmatrix} 0 & A & 0 \\ K & C & D \\ 0 & L & 0 \end{pmatrix},$$

where K and L are compact operators. As in Theorem 3.2 we can find a factorization

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \begin{pmatrix} U & V \end{pmatrix}.$$

It is easy to see that we can write $K = K_1K_2K_3$ and $L = L_1L_2L_3$, where all operators K_i and L_i are compact. Define operators

$$M = \begin{pmatrix} 0 & 0 & X \\ K_1 & 0 & Y \\ 0 & L_1 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & K_2 & 0 \\ 0 & 0 & L_2 \\ I & 0 & 0 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0 & U & V \\ K_3 & 0 & 0 \\ 0 & L_3 & 0 \end{pmatrix}.$$

It is clear that $S = MNP$ and M^2, N^2, P^2 are compact, which completes the proof. □

Theorem 4.2. *Let T be an operator on $\mathcal{H}$. The following assertions are equivalent*:

(a) *The operator T is similar to an operator S with $0 \in \sigma_e(SS^* + S^*S)$.*
(b) *$\mathcal{H}$ can be decomposed as a direct sum of three infinite-dimensional closed subspaces, so that T is similar to an operator of the form*

$$\begin{pmatrix} 0 & K_1 & 0 \\ K_2 & C & K_3 \\ 0 & K_4 & 0 \end{pmatrix},$$

where K_i are compact operators.
(c) *The operator T is a product of two quasi-nilpotent operators whose squares are compact.*
(d) *The operator T is a product of two operators whose squares are compact.*

Proof. For the proof of implications from (a) to (b) and from (b) to (c) we just slightly change the proof of Theorem 4.1. Suppose first that T is similar to an operator S with $0 \in \sigma_e(SS^* + S^*S)$. We can find decomposition of the space $\mathcal{H}$ as before. In addition, since $0 \in \sigma_e(SS^* + S^*S)$ we can choose a sequence $f_1, g_1, f_2, g_2, \ldots$ such that $\|Sf_n\| \leqslant \frac{1}{n}$, $\|Sg_n\| \leqslant \frac{1}{n}$ and the same holds for S^*. Then the matrix of S relative to this decomposition is

$$\begin{pmatrix} K_1 & A & B \\ K_2 & C & D \\ K_3 & K_4 & K_5 \end{pmatrix},$$

where also A, B and D are Hilbert–Schmidt operators and therefore compact. The rest of the proof is the same as before. Since A and D are compact, the squares of both operators in the factorization are compact. The direct proof of implication form (a) to (c) can be found in [4].

Since the implication from (c) to (d) is obvious, the only thing left to prove is that (d) implies (a). Assume that $T = MN$, where M^2 and N^2 are compact. If T is compact, then it is clear that $0 \in \sigma_e(TT^* + T^*T)$. So suppose that T is not compact. We use the known characterization of compact operators (see, e.g. [3]): an operator K is compact if and only if for every orthonormal sequence $\{e_n\}$ the sequence $\{Ke_n\}$ converges to zero. Since T is not compact, there exists an orthonormal sequence $\{e_n\}$ such that $\{Te_n\}$ does not converge to zero. Define $f_n = Ne_n$ and $g_n = M^*Te_n$. The sequences $\{f_n\}$ and $\{g_n\}$ converge weakly to zero, $\|Tf_n\| \to 0$, $\|T^*g_n\| \to 0$ and $\langle f_n, g_n\rangle \nrightarrow 0$. By [4, Lemma 4] the operator T is similar to an operator S with $0 \in \sigma_e(SS^* + S^*S)$. This completes the proof. □

Acknowledgment

The author would like to thank Roman Drnovšek for many helpful suggestions during the preparation of the paper.

References

[1] R.G. Douglas, On majorization, factorization, and range of inclusion of operators on Hilbert space, Proc. Amer. Math. Soc. 17 (1966) 413–415.
[2] R. Drnovšek, V. Müller, N. Novak, An operator is a product of two quasi-nilpotent operators if and only if it is not semi-Fredholm, Proc. Roy. Soc. Edinburgh Sect. A 136 (2006) 935–944.
[3] P.A. Fillmore, J.P. Williams, On operator ranges, Adv. Math. 7 (1971) 254–281.
[4] C.K. Fong, A.R. Sourour, Sums and products of quasi-nilpotent operators, Proc. Roy. Soc. Edinburgh Sect. A 99 (1984) 193–200.
[5] T.J. Laffey, Products of matrices, in: Generators and Relations in Groups and Geometries, in: NATO ASI Series, Kluwer, 1991.
[6] I.E. Schochetman, R.L. Smith, S.-K. Tsui, On the closure of the sum of closed subspaces, Int. J. Math. Math. Sci. 26 (5) (2001) 257–267.
[7] A.R. Sourour, Nilpotent factorization of matrices, Linear Multilinear Algebra 31 (1992) 303–309.
[8] R.P. Sullivan, Products of nilpotent linear transformations, Proc. Roy. Soc. Edinburgh Sect. A 124 (1994) 1135–1150.
[9] P.Y. Wu, Products of nilpotent matrices, Linear Algebra Appl. 96 (1987) 227–232.